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Periodic Solutions of a Forced Lotka–Volterra Equation*

P. TÁBOAS

ICMSC-USP, Caixa Postal 668,
13.560–São Carlos–SP, Brazil

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For a 2-species predator–prey model, the effect of a periodic forcing dependent on two parameters is studied. A description is given for the local bifurcation curves of periodic solutions near a fixed family of solutions of the free equation. © 1987 Academic Press, Inc.

INTRODUCTION

This work is an application of an artifice created by Hale and the author [8] in the investigation of bifurcations of periodic solutions near a family of solutions of a second-order ordinary differential equation. This kind of problem has attracted attention in recent years, the references [1, 2, 5, 7, 13] allude to the problem of local bifurcation near families of critical solutions.

In spite of the fact that [8] just shows an application to the nonlinear oscillator, it goes beyond that, since the ideas there lead to a formulation in an abstract setting, see [5, 7], for example, or can be transposed to distinct particular situations, see [2, Chap. 11, 3].

The disposition assumed here is related to this last alternative. The concern is a periodically forced Lotka–Volterra model for a 2-species predator–prey system

$$\begin{aligned} \dot{x}_1 &= ax_1 - bx_1x_2 + \varepsilon_1 f_1(t, x_1, x_2) \\ \dot{x}_2 &= cx_1x_2 - dx_2 + \varepsilon_2 f_2(t, x_1, x_2) \end{aligned} \tag{0.1}$$

where $a, b, c, d > 0$ are constants, $x_1, x_2 \geqslant 0$ are the prey and predator populations, respectively, $\varepsilon = (\varepsilon_1, \varepsilon_2)$ is a real vector parameter and

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$f_j(t+T, x_1, x_2) = f_j(t, x_1, x_2)$, $j=1, 2$; $(t, x_1, x_2) \in R^3$, for a constant $T>0$. It is known that, for $\varepsilon=0$, the singular point $(x_1, x_2) = (d/c, a/b)$ is a center. Suppose that $e(t) = (e_1(t), e_2(t))$ is a T-periodic solution of (0.1). The objective is to answer the following questions: How many T-periodic solutions of (0.1) are near e_α, $e_\alpha(t) = e(t+\alpha)$, for some α, $0 \leqslant \alpha < T$, when ε remains in a neighborhood of the origin in the ε-plane? How does the number of these solutions vary when ε runs over that neighborhood?

These are precisely the same questions implicitly asked in [8] for the nonlinear oscillator, but the present problem has some special features.

In [8], one of the parameters measures a dissipation of energy, so that when the other parameter is zero, there is no periodic solution of the perturbed system. This fact implies that always bifurcations occur in a neighborhood of $\varepsilon=0$. That is not the case here. There are conditions under which (0.1) has precisely two T-periodic solutions near some e_α, $0 \leqslant \alpha < T$, for $|\varepsilon|$ small, see corollary to Theorem 2.

The linear variational problem here is not self-adjoint, while in [8] that is precisely the case.

The technique employed is a nonstandard application of the Liapunov–Schmidt reduction. In general, when this method is applied in the usual way for a 2-point boundary value problem, it leads to an equation in the null-space of a Fredholm operator in a Banach space of solutions. The independent variable becomes the component of the solutions in this null-space, which in the present problem is 1-dimensional. A convenient choice of the projections involved with the method suppresses the coordinate in the null-space, and its role is played by the phase shift α.

Finally, it should be said that the effect of a periodic forcing in the Lotka–Volterra equation was studied by Hausrath in [9], where the Hamiltonian character of (0.1) is exploited in order to state the existence of periodic integral manifolds of (0.1), where the forcing has the form $f_j(t, x_1, x_2) = x_j g(t, x_1, x_2)$, $j=1, 2$.

1. Formulation of the Problem and Statement of Results

The Lotka–Volterra equation for a 2-species predator–prey system is

$$\dot{x} = g(x) \tag{1.1}$$

$x = (x_1, x_2)$, $g(x) = (ax_1 - bx_1x_2,\ cx_1x_2 - dx_2)$, where a, b, c, d are positive constants, the real components x_1, x_2 are the biomass of the prey and predator species, with a and d being their natural growth and decline rates, respectively, while b and c define effects of predation.

Some well-known facts mentioned below are tacitly admitted throughout this paper.

The meaningful first quadrant $Q_1 = \{x: x_1, x_2 > 0\}$ is invariant, so that Eq. (1.1) will be understood just in Q_1. Every orbit in Q_1 is periodic and encircles the center $x_0 = (d/c, a/b)$. There are smooth functions $\varphi: R \times J \to Q_1$, $\omega: J \to J$, where $J = [0, \infty)$; $\varphi(\theta + 1, \rho) = \varphi(\theta, \rho)$, $(\theta, \rho) \in R \times J$, such that the general solution of (1.1) is given by $x(t) = \varphi(\omega(\rho)(t + \alpha), \rho)$, where ρ is the amplitude and α the phase.

Let $e(t) = \varphi(\omega(\rho_0)\, t, \rho_0) = (e_1(t), e_2(t))$ be a fixed T-periodic solution $T = [\omega(\rho_0)]^{-1}$, $\rho_0 > 0$. This work is concerned with the appearance of T-periodic orbits near $\Gamma = \{e(t): 0 \leqslant t < T\}$ under the action of T-periodic small forcing terms.

In a precise setting, regarding to the perturbed equation

$$\dot{x} = g(x) + \hat{\varepsilon} f(t, x) \tag{1.2}$$

where $\hat{\varepsilon}$ is a 2×2 diagonal real matrix $\hat{\varepsilon} = \operatorname{diag}(\varepsilon_1, \varepsilon_2)$; $f = (f_1, f_2)$ is a function of class C^2 such that $f(t + T, x) = f(t, x)$, $(t, x) \in R \times Q_1$, the main purpose is to characterize the T-periodic solutions of (1.2), which are near $e_\alpha(t) = e(t + \alpha)$, for some $\alpha \in [0, T)$, in the sense of a C^1-norm, with $\varepsilon = (\varepsilon_1, \varepsilon_2)$ ranging over a neighborhood of the origin in the ε-plane.

One of the hypotheses needed is

(H1) $$\omega'(\rho_0) \neq 0.$$

The results by Hsu in [10] imply that the period T is a strictly increasing function of the amplitude ρ of the periodic orbit, and $\lim_{\rho \to +\infty} \tau(\rho) = +\infty$. This shows that the set of $\rho_0 > 0$ in which (H1) is satisfied is arbitrarily large. Indeed, (H1) is likely true for any $\rho_0 > 0$.

Hypothesis (H1) says that $\dot{e}$ spans the space of the T-periodic solutions of the linear variational equation of (1.1) around e,

$$\dot{x} = g'(e(t))\, x \tag{1.3}$$

where

$$g'(e(t)) = \begin{bmatrix} a - be_2(t) & -be_1(t) \\ ce_2(t) & ce_1(t) - d \end{bmatrix}.$$

Actually, if $\delta(t) = (\partial/\partial\rho)[\psi(\omega(\rho)\, t, \rho)]_{\rho = \rho_0} = (\dot{e}(t)/\omega(\rho_0))\, \omega'(\rho_0)\, t + r(t)$, where $r(t) = (\partial\phi/\partial\rho)(\omega(\rho_0)\, t, \rho_0)$ is a T-periodic function, it is easily seen that δ is another solution of (1.3), which is unbounded and, therefore, linearly independent of $\dot{e}$.

The assumptions on f are

(H2) $$f(t, x) = h(t) + k(t, x).$$

Where h and k are R^2-valued function of class C^2, h is T-periodic and

k is T-periodic in t, for each fixed $x \in R^2$, and satisfies the smallness conditions

$$k(t, e(\tau)) = \frac{\partial k(t, e(\tau))}{\partial x_j} = 0, \qquad t \in R, 0 \leqslant \tau < T, j = 1, 2.$$

A forcing f independent of x, i.e., $k(t, x) \equiv 0$, is probably the most important case covered by the hypothesis (H2). Physically, it can be interpreted as a T-periodic hunting which is independent of the size of the populations. Indeed, since this kind of forcing is not included in [9], it was part of the initial motivation of the present paper. However, the hypothesis (H2) defines a wider class of admissible forcing terms.

It means that if the populations deviate not much from Γ, the effect of the forcing f depends more on the time than on the deviation.

In the next section the reduction of Liapunov and Schmidt will be applied to lead to the bifurcation equation

$$G(\varepsilon, \alpha) = 0 \tag{1.4}$$

where α is a real parameter which characterizes the phase of a T-periodic solution of (1.2). The coefficients of the linear terms in the expansion of G around $\varepsilon = 0$ are the T-periodic functions of α,

$$v_1(\alpha) = d \int_0^T \left[\frac{h_1(t+\alpha)}{e_1(t)}\right] dt - cM_1$$

$$v_2(\alpha) = a \int_0^T \left[\frac{h_2(t+\alpha)}{e_2(t)}\right] dt - bM_2$$

with $M_j = \int_0^T h_j(t)\, dt$, $j = 1, 2$.

For the closed curve $v(\alpha) = (v_1(\alpha), v_2(\alpha))$, $0 \leqslant \alpha < T$, the following hypotheses are assumed:

(H3) *$v(\alpha)$, $v'(\alpha)$ are linearly independent, except possibly for finitely many values $\alpha = \alpha_j$, $j = 1, \ldots, n$, with $v(\alpha_i)$, $v(\alpha_j)$ linearly independent, if $i \neq j$. $v''(\alpha)$ is never collinear with both, $v(\alpha)$ and $v'(\alpha)$, $0 \leqslant \alpha < T$.*

(H4) $$v(\alpha) \neq 0, 0 \leqslant \alpha < T.$$

Hypothesis (H3) says that the plane curve $\{v(\alpha), \alpha \in [0, T]\}$ defined by the function v can be tangent to at most a finite number of lines through the origin and, in this case, the tangency point $v(\bar{\alpha})$ is unique. The nondegeneracy condition on $v''(\bar{\alpha})$ imply that $v(\alpha)$ lies in one of the semi-planes determined by the tangent line, for α near $\bar{\alpha}$. See Fig. 1.1, where the tangent line is represented by the dotted line.

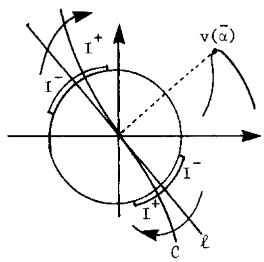

FIG. 1.1. Increasing direction of crossing l and C.

Knowing the solution e, the hypothesis (H3) has only to do with the T-periodic part h of the perturbation f. As the negation of (H3) is sensitive to small perturbations of h, one can say that only exceptionally this hypothesis does not hold. The corollary to Theorem 2 below gives conditions on h to ensure that the curve defined by v is a circle centered at the origin. In this case (H3) is satisfied with $n = 0$.

Some restrictions like those of hypotheses (H3), (H4) must be imposed on $v(\alpha)$, in order to make possible a systematic investigation of the bifurcation equation. However, a careful analysis shows that the problem can be done under weaker assumptions with little apparent gain.

The introduction of some appropriate terminology is necessary to make more specific the statement of the results.

Suppose (H3), (H4) are satisfied and $\bar{\alpha}$, $0 \leqslant \bar{\alpha} < T$, is such that $v(\bar{\alpha})$ and $v'(\bar{\alpha})$ are collinear. Let l be the line $\varepsilon \cdot v(\bar{\alpha}) = 0$. Centered in $\varepsilon_0 \in l$, $|\varepsilon_0| = \beta \neq 0$, consider a small arc I of the circle C_β: $|\varepsilon| = \beta$. I is divided by ε_0 into two arcs, I^+ and I^-, in such a way that $\varepsilon \in I^-$, if $[\varepsilon \cdot v(\alpha)][\varepsilon_0 \cdot v(\alpha)] \geqslant 0$, for α in a suitable neighborhood of $\bar{\alpha}$, otherwise $\varepsilon \in I^+$. When l is crossed by moving on C_β from I^- to I^+, it is said that l is *crossed in the increasing direction.* This also defines the increasing direction of crossing for a curve $\mathscr{C}$ at a point $\varepsilon_0 \in \mathscr{C}$, with $|\varepsilon_0|$ sufficiently small, when $\mathscr{C}$ is tangent to l at $\varepsilon = 0$. In the Fig. 1.1 the arrows indicate the increasing direction of crossing l and $\mathscr{C}$, near ε_0 and $-\varepsilon_0$.

Before stating the results, another definition is needed. If the hypotheses (H3), H(4) are satisfied, for a sufficiently small ball V, with center $\varepsilon = 0$, the set $\Sigma \subset V$ is defined by

$$\Sigma = \{\varepsilon \in V \colon \varepsilon \cdot v(\alpha) = 0, \text{ for some } \alpha \in [0, T)\}.$$

Either $\Sigma = V$ or the boundary of Σ in V is $(l_1 \cup l_2) \cap V$, where l_1, l_2 are the lines $\varepsilon \cdot v(\alpha_1) = 0$, $\varepsilon \cdot v(\alpha_2) = 0$, respectively, where α_1, α_2 belonging to the set $\{\alpha_j\}$ in hypothesis (H3) are uniquely defined by the property that the lines through $v(\alpha_1)$, $v(\alpha_2)$ intersect the curve defined by v only at tangencies.

The set S considered in the statement of the Theorem 1 is a small perturbation of Σ. Let X denote the Banach space of the R^2-valued T-periodic functions of class C^1, with a C^1-norm.

THEOREM 1. *If the hypotheses* (H1)–(H4) *are satisfied, then there exists a neighborhood* $U \subset X$ *of* e_α, $0 \leqslant \alpha < T$, *and a ball* $V \subset R^2$, *centered at* $\varepsilon = 0$, *such that the following conditions are fulfilled*:

(A) *For each* $\bar{\alpha}$, *with* $v(\bar{\alpha})$, $v'(\bar{\alpha})$ *collinear, there exists a curve* C, *tangent to the line* $\varepsilon \cdot v(\bar{\alpha}) = 0$ *at* $\varepsilon = 0$, *which divides* V *into two connected components, such that the number of solutions of* (1.2) *in* U *increases precisely by two when* ε *crosses* C *in the increasing direction.*

(B) *There exists a set* $S \subset V$ *such that, if* $\varepsilon \in S$, *there are at least two solutions of* (1.2) *in* U *and, if* $\varepsilon \in V \backslash S$, *there are no solutions of* (1.2) *in* U. *The set* S *is defined as follows: If* $\Sigma = V$, *then* $S = V$. *If the boundary of* Σ *in* V *is* $(l_1 \cup l_2) \cap V$, *then the curves* C_1, C_2 *corresponding to* α_1, α_2, *respectively, in part* (A), *divide* V *into four connected components. The set* S *is the union of the two components such that, crossing* C_1 *or* C_2 *in the increasing direction corresponds to crossing from outside to inside* S.

Remark. By using polar coordinates $v(\alpha) = \rho(\alpha)(\cos\theta(\alpha), \sin\theta(\alpha))$, $0 \leqslant \theta(\alpha) < 2\pi$, $0 \leqslant \alpha < T$, it could happen that, for two values $\bar{\alpha}$, $\underline{\alpha}$ among those for which $v(\alpha)$, $v'(\alpha)$ are collinear, the argument $\theta(\alpha)$ satisfies $\theta(\underline{\alpha}) \leqslant \theta(\alpha) \leqslant \theta(\bar{\alpha})$, $0 \leqslant \alpha < T$. If $\theta(\bar{\alpha}) - \theta(\underline{\alpha}) < \pi$ [Fig. 1.2], the curves $\underline{\mathscr{C}}$ and $\bar{\mathscr{C}}$, related to $\underline{\alpha}$ and $\bar{\alpha}$ in the Theorem 1, determine the boundary of the region S. If either $\theta(\bar{\alpha}) - \theta(\underline{\alpha}) > \pi$ [Fig. 1.3] or there are no such values $\underline{\alpha}$, $\bar{\alpha}$, the region S becomes the full neighborhood V. The shaded parts of Figs. 1.2–1.4 show the region S in some possible situations.

The number in parenthesis shows the number of solutions in U for ε in that region.

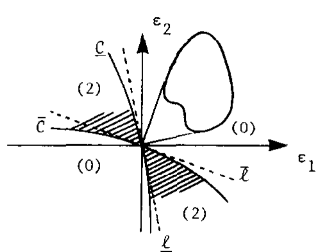

FIG. 1.2. S is a sectorial region.

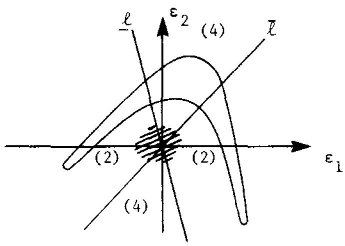

FIG. 1.3. $S = V$, with bifurcation.

THEOREM 2. *If the hypotheses* (H1)–(H3) *are satisfied with* $v(\alpha)$ *and* $v'(\alpha)$ *linearly independent,* $0 \leqslant \alpha < T$, *then there exists a neighborhood* $U \subset X$ *of* e, *and a neighborhood* $V \subset R^2$ *of* $\varepsilon = 0$, *such that for each* ε *in* V, *the Equation* (1.2) *has precisely two T-periodic solutions in* U.

Remark. Under the hypotheses of the Theorem 2, there is no bifurcation of T-periodic solution of Eq. (1.2) in U. This is indicated in Fig. 1.4.

If r is a continuous complex-valued function of R/TZ, $r(n)$ denotes the nth Fourier coefficient of r with respect to $e^{i\omega n t}$, $\omega = 2\pi/T$, $n \in Z$, that is, $\hat{r}(n) = (1/T) \int_0^T e^{i\omega n t} r(t)\, dt$.

Letting $r_1(t) = 1/e_1(t)$, $r_2(t) = 1/e_2(t)$, since r_1, r_2 can be regarded as a nonconstant positive function on R/TZ, it follows that $\hat{r}_1(n)$, $\hat{r}_2(n)$, $n = -1, 1$, are nonzero complex numbers.

The following corollary provides an example of application of the Theorem 2.

COROLLARY. *Suppose that hypotheses* (H1), (H2) *are satisfied, where the real function h in* (H2) *is given by*

$$(h_1(t), h_2(t)) = (a_{-1} e^{-i\omega t} + a_1 e^{i\omega t}, b_{-1} e^{-i\omega t} + b_1 e^{i\omega t}),$$

with $a_n = [2Td\hat{r}_1(n)]^{-1}$, $b_n = -i[2Ta\hat{r}_2(n)]^{-1}$, $n = -1, 1$. *Then, there exist a neighborhood* $U \subset X$ *of* e, *and a neighborhood* $V \subset R^2$ *of* $\varepsilon = 0$, *such that the* Eq. (1.2) *has precisely two T-periodic solutions in* U, *for each* $\varepsilon \in V$.

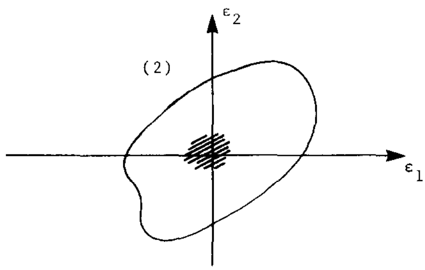

FIG. 1.4. $S = V$, with no bifurcation.

Remark. It will be shown in Section 4 that, under the assumptions of this corollary, the curve $v(\alpha)$, $0 \leqslant \alpha < T$, is the circle $(\cos \omega\alpha, \sin \omega\alpha)$. Actually, the proof of this corollary suggests a device for solving, at least in particular cases, the interesting inverse problem.

Given a finite pencil of lines through the point $\varepsilon = 0$, $l_1, ..., l_n$, find a forcing f which satisfies the hypothesis (H2), in such a way that the diagram of bifurcation of T-periodic solutions of (1.2) in U is determined by curves $\mathscr{C}_1, ..., \mathscr{C}_n$, tangent to lines $l_1^*, ..., l_n^*$, at $\varepsilon = 0$, whose slopes are approximately those of $l_1, ..., l_n$, respectively, at $\varepsilon = 0$, according to the description of the Theorem 1.

A desirable behavior of the populations (e.g., a large number of T-periodic solutions in U, with ε under natural bounds) can be stated a priori in many practical situations. Then, the problem is "how" should be the hunting or some other external agent (this is determined by the forcing) in order to obtain that behavior. This comment leads to the conclusion that, for the applications, in many situations this inverse problem should be the question of greater interest.

One should be naturally concerned with the description of the limiting behavior of the solutions of (1.2) in U, as ε approaches zero from the interior of S. Indeed, these solutions can be regarded as bifurcating from the family $\{e(t+\alpha)\}$, $0 \leqslant \alpha < T$, when the parameter ε crosses the origin entering the region S. The following theorem provides a satisfactory answer to this question. It is an adaption of [8, Theorem 1.2], but here a hypothesis of uniform boundedness is eliminated.

Some notation, suggested by [2, p. 375], is needed.

If hypothesis (H1)–(H4) are satisfied, the phrase "a curve approaching zero" means a continuous curve in the ε-plane: $\gamma: \varepsilon = \varepsilon(\beta) \in S$, $0 \leqslant \beta \leqslant 1$, with $\varepsilon(\beta) = 0$ if and only if $\beta = 0$. Given such a curve γ, $\varepsilon(\beta)$ in polar coordinates $\rho(\beta)(\cos \theta(\beta), \sin \theta(\beta))$, $0 < \beta \leqslant 1$, defines $\underline{\theta} = \liminf_{\beta \to 0} \theta(\beta)$ and $\bar{\theta} = \limsup_{\beta \to 0} \theta(\beta)$. In the sequel the interval $[0, T)$ is regarded as a representation of the quotient topological space R/TZ and the sets $U \subset X$ and $V \subset R^2$ are the neighborhoods given in the Theorem 1.

THEOREM 3. *Suppose the hypotheses* (H1)–(H4) *are satisfied. Let* $\gamma: \varepsilon = \varepsilon(\beta)$, $0 \leqslant \beta \leqslant 1$, *be a continuous curve approaching zero, and* $x(\cdot\,; \beta) \in U$ *be a solution of* (1.2) *for* $\varepsilon = \varepsilon(\beta)$, *which depends continuously on* β. *Then* $\mathscr{S} = \{x(\cdot\,; \beta) \in X, 0 \leqslant \beta \leqslant 1\}$ *is precompact and every limit point of* $\mathscr{S}$, *as* $\beta \to 0$, *is a T-periodic solution of* (1.1). *Moreover, if* $\bar{\theta} > \underline{\theta}$, *there exists a nondegenerate interval* $I(\gamma) = [\underline{\alpha}, \bar{\alpha}] \subset [0, T)$, *such that* $\alpha \in I(\gamma)$ *implies* $v(\alpha) \cdot \varepsilon = 0$, *for some* $e = \rho(\cos \theta, \sin \theta)$, *with* $\underline{\theta} \leqslant \theta \leqslant \bar{\theta}$ *and, finally*

$$\mathrm{Cl}(\mathscr{S}) \backslash S = \{e_\alpha \in X: \alpha \in I(\gamma)\}$$

where $e_\alpha(t) = e(t - \alpha)$.

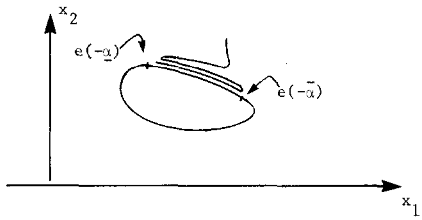

FIG. 1.5. Behavior of the initial values, $x(0, \beta)$, as $\beta \to 0$.

Remark. The Fig. 1.5 shows how the curve of initial values $x(0, \beta)$ could behave near the arc $J = \{e_\alpha(0) \in R^2 \colon \alpha \in I(\gamma)\}$ of Γ.

2. The Bifurcation Equation

In order to expand the vector field of (1.2) around e, a tubular neighborhood of the orbit Γ is needed. Let $e^\perp(\alpha)$ be the orthogonal vector $e^\perp = (\dot{e}_2(\alpha), -\dot{e}_1(\alpha))$, $0 \leqslant \alpha < T$, so that each point x in a suitable neighborhood W of Γ can be associated to coordinates (τ, σ), $0 \leqslant \tau < T$, $|\sigma| < \sigma_0$, $[\sigma_0 > 0$, a constant$]$ defined by the equation

$$x = e(\tau) + \sigma e^\perp(\tau). \tag{2.1}$$

For each T-periodic solutions $x(t)$ of (1.2) which remains in W, an α, $0 \leqslant \alpha < T$, can be uniquely chosen in such a way that the coordinates of $x(\alpha)$ given by (2.1) are of the form $(\tau, \sigma) = (0, \sigma)$. Thus, such a solution can be represented by

$$x(t) = e(t - \alpha) + z(t - \alpha), \qquad z(0) \cdot \dot{e}(0) = 0. \tag{2.2}$$

The parameter α can be associated with the phase of the solutions, since $x(\alpha)$ is near $e(0)$.

For each α, $0 \leqslant \alpha < T$, the 1–1 correspondence between solutions $x(t)$ of (1.2) and $x(t + \alpha)$ of

$$\dot{x} = g(x) + \hat{\varepsilon} f(t + \alpha, x) \tag{2.3}$$

makes the problem equivalent to studying the T-periodic solutions of (2.3) of the form

$$x(t) = e(t) + z(t), \qquad z(0) \cdot \dot{e}(0) = 0 \tag{2.4}$$

with small z. The relation (2.4) substituted into Eq. (2.3) leads to the search for small T-periodic solutions z of

$$\begin{aligned} \dot{z} &= g'(e(t))\, z + \hat{\varepsilon} h_\alpha(t) + \mathscr{R}(z) \\ z(0) \cdot \dot{e}(0) &= 0 \end{aligned} \tag{2.5}$$

where $h_\alpha(t) = h(t+\alpha)$, the function $h = (h_1, h_2)$ is the part of f given in (H2), and $\mathscr{R}(z) = O(|z|^2)$, as $z \to 0$.

By using the first integral of the Eq. (1.1), $U(x) = a \ln x_2 + d \ln x_1 - cx_1 - bx_2$, it can be seen that $\eta(t) = (d/e_1(t) - c,\ a/e_2(t) - b)$ is a T-periodic solution of the adjoint of the linear variational equation of (1.1) around $e(t)$

$$\dot{z} + [g'(e(t))]^* z = 0 \tag{2.6}$$

where A^* denotes the adjoint of A. See Hale [6, p. 283], for instance. As a consequence of (H1), the space of T-periodic solutions of (2.6) must be 1-dimensional and, therefore, spanned by the solution η.

Let Z be the Banach space of the continuous T-periodic R^2-valued functions with the supremum norm and X, as before, its algebraic subspace of the functions of class C^1, with a C^1 norm.

The maps $P\colon X \to X$, $Q\colon Z \to Z$ are the projections given by

$$\begin{aligned} P\varphi &= |\dot{e}(0)|^{-2} (\varphi(0) \cdot \dot{e}(0))\, \dot{e}, \qquad && \varphi \in X \\ Q\varphi &= \nu \left[\int_0^T \eta(t) \cdot \varphi(t)\, dt \right] \eta, \qquad && \varphi \in Z \end{aligned}$$

where

$$\nu = \left[\int_0^T \eta^2(t)\, dt \right]^{-1}.$$

In this setting, if the maps $L\colon X \to Z$, $N\colon X \times R^3 \to Z$ are defined by $Lz = \dot{z} - g'(e)\, z$, $N(z, \varepsilon, \alpha) = \hat{\varepsilon} h_\alpha + \mathscr{R}(z)$, the problem (2.5) can be compactly written as a system of equations in X:

$$\begin{aligned} &\text{(a)} \qquad Lz = N(z, \varepsilon, \alpha) \\ &\text{(b)} \qquad Pz = 0 \end{aligned} \tag{2.7}$$

and, if $\mathscr{R}(L)$ and $\mathscr{N}(L)$ denote the range and the null space of the linear map L, a *Fredholm alternative* type statement can be formulated as the following lemma.

LEMMA 2.1. $\mathscr{R}(L) = (I - Q) Z$.

Proof. A T-periodic function $\varphi \in Z$ is in $\mathscr{R}(L)$ if and only if the ordinary differential equation $Lz = \varphi$ has a T-periodic solution. But it is well known that this is equivalent to the condition $\int_0^T \eta(t) \cdot \varphi(t)\, dt = 0$, since η spans the T-periodic solutions of the adjoint equation (2.6). See Hale [6, p. 146], for instance. The last condition means that $Q\varphi = 0$ and therefore, completes the proof.

If $\varphi \in (I - Q) Z$, there exists a unique solution $z = K\varphi \in X$ of the equation $Lz = \varphi$ such that $Pz = 0$. In fact, let $z_0(\varphi) \in X$ be a particular solution of $Lz = \varphi$, so that its general solution $z \in X$ is given by $z(t) = z_0(\varphi)(t) + \beta\dot{e}(t)$, $\beta \in R$. The condition $z(0) \cdot e(0) = 0$ uniquely defines β.

The choice of a C^1-norm in X ensures the continuity of the linear map L. Moreover, the restriction of L to the closed subspace $(I - P) X$ is a bijection from $(I - P) X$ to $\mathscr{R}(L)$, whose inverse is precisely the operator K. It is now a consequence of the closed graph theorem that K is continuous. See Nirenberg [12, Theorem 2, Chap. VI], for instance. So that K is a continuous right inverse of L, i.e., $LK = I$, the identity in $\mathscr{R}(L)$, and $KL = I - P$ in X.

The reduction of Liapunov and Schmidt is now applied to Eq. (2.7)(a). That is, this equation is separated in its components in the supplementary subspaces QZ and $(I - Q) Z$. By taking into account the properties of the operator K, these components can be written as

$$\begin{aligned} z &= Pz + K(I - Q)\, N(z, \varepsilon, \alpha) \\ 0 &= QN(z, \varepsilon, \alpha) \end{aligned} \tag{2.8}$$

and, by reason of Eq. (2.7)(b), the problem (2.7) is equivalent to

$$\begin{aligned} &\text{(a)} \qquad z = K(I - Q)\, N(z, \varepsilon, \alpha) \\ &\text{(b)} \qquad 0 = QN(z, \varepsilon, \alpha) \end{aligned} \tag{2.9}$$

The compactness of the interval $0 \leqslant \alpha \leqslant T$ implies, after a finite number of applications of the implicit function theorem (see [11, Part 2, VI, Sect. 2], for instance) that there exists a neighborhood $U_0 \subset X$ of $z = 0$ and a neighborhood $V \subset R^2$ of $\varepsilon = 0$, such that Eq. (2.9)(a) has a solution $z = z^*(\varepsilon, \alpha)$, $\varepsilon \in V$, $0 \leqslant \alpha \leqslant T$, which is of class C^2, is unique in U_0 and satisfies $z^*(0, \varepsilon) = 0$, $0 \leqslant \alpha \leqslant T$.

Therefore, for any $\varepsilon \in V$, there exists a T-periodic solution of (1.2) in the neighborhood U of e_α, $\alpha \in [0, T]$, in X, which can be represented by (2.2), if and only if, Eq. (2.9)(b), with z replaced $z^*(\varepsilon, \alpha)$, is satisfied by (ε, α). In other words, this is to say that (ε, α) satisfies the bifurcation equation

$$v \int_0^T \eta(t)[\hat{\varepsilon} h_\alpha(t) + (z^*(\varepsilon, \alpha))(t)]\, dt = 0 \tag{2.10}$$

which, according to the expression for η, can be rewritten as

$$\left[d\int_0^T \frac{h_1(t+\alpha)}{e_1(t)}\,dt - cM_1\right]\varepsilon_1 + \left[a\int_0^T \frac{h_2(t+\alpha)}{e_2(t)}\,dt - bM_2\right]\varepsilon_2 + S(\varepsilon,\alpha) = 0 \tag{2.11}$$

where $S(\varepsilon, \alpha) = O(|\varepsilon|^2)$, as $\varepsilon \to 0$.

3. Proof of the Theorem 1

According to the Section 2, the proof reduces to the investigation of the solutions (ε, α) of Eq. (2.11), which is now written in the compact form

$$v(\alpha)\cdot\varepsilon + S(\varepsilon,\alpha) = 0. \tag{3.1}$$

For $\varepsilon \neq 0$, let $\varepsilon = ru$, where u is a unit vector and $r = |\varepsilon|$. Then (3.1) is equivalent to

$$F(r, u, \alpha) = v(\alpha)\cdot u + V(r, u, \alpha) = 0 \tag{3.2}$$

where $V(r, u, \alpha) = (1/r)\,S(ru, \alpha)$. Therefore, since $V(0, u, \alpha) \equiv 0$, the only solutions for $r = 0$ are those $(0, u_0, \alpha_0)$ such that u_0 and $v(\alpha_0)$ are orthogonal. If u_0, v_0 are chosen in this way, an extension of $F(r, u, \alpha)$ to a full neighborhood of $(0, u_0, \alpha_0)$ in $R\times S^1\times R$ can be made by defining $V(r, u, \alpha) = -(1/r)\,S(-ru, \alpha)$, for $r<0$, $|r|$, $|u-u_0|$, $|\alpha-\alpha_0| < \delta_0$, $\delta_0 > 0$ a constant.

There are two cases to be analysed:

First, suppose that $v(\alpha_0)$ and $v'(\alpha_0)$ are linearly independent. Therefore, $\partial F(0, u_0, \alpha_0)/\partial\alpha = v'(\alpha_0)\cdot u_0 \neq 0$, and an application of the implicit function theorem shows there exists a number $\delta > 0$ such that, for $(r, u) \in R\times S^1$, $0 \leqslant r < \delta$, $|u-u_0| < \delta$, there exists $\alpha^*(r, u)$, with α^* being a function of class C^1 such that $F(r, u, \alpha^*(r, u)) = 0$. Moreover, $(r, u, \alpha^*(r, u))$ is the only solution of (3.2) in a suitable neighborhood of $(r, u, \alpha) = (0, u_0, \alpha_0)$. So that there is no bifurcation of solution of (3.2) near $(0, u_0, \alpha_0)$.

Second, suppose that $v(\alpha_0)$ and $v'(\alpha_0)$ are collinear. According to the hypothesis (H3), $v''(\alpha_0)$ and $v'(\alpha_0)$ are linearly independent. Therefore, $\partial^2 F(0, u_0, \alpha_0)/\partial\alpha^2 = v''(\alpha_0)\cdot u_0 \neq 0$ and, once more by an application of the implicit function theorem, there exists a number $\delta > 0$ such that, for $(r, u) \in R\times S^1$, $0 \leqslant r < \delta$, $|u-u_0| < \delta$, there exists a unique $\alpha^*(r, u)$, with α^* being a function of class C^1 such that $\partial F(r, u, \alpha^*(r, u))/\partial\alpha = 0$, $\alpha^*(0, u_0) = \alpha_0$. Therefore, $M(r, u) = F(r, u, \alpha^*(r, u))$ is a maximum value of

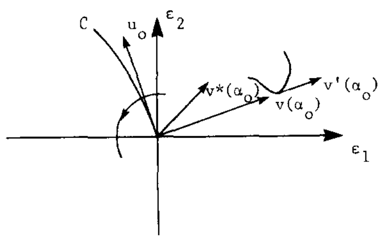

FIG. 3.1. Case in which $v''(x_0) \cdot v_0 > 0$.

the function $F(r, u, \cdot)$ according to $v''(\alpha_0) \cdot u_0$ being negative or positive, respectively. Elementary calculations show that, for u in the unit sphere S^1, $\partial M(0, u_0)/\partial u$ is the inner product of $v(\alpha_0)$ and a unit vector orthogonal to u_0. Since $v(\alpha_0) \neq 0$ is collinear with such a vector, it is seen that $\partial M(0, u_0)/\partial u \neq 0$. Therefore, the implicit function theorem ensures that the equation $M(r, u) = 0$ has a unique continuously differentiable solution $u = u^*(r)$, $0 \leqslant r < \bar{\bar{\delta}}$, $u^*(0) = u_0$, $\bar{\bar{\delta}} > 0$, a constant. This solution describes a curve $\mathscr{C}$ in the ε-plane, which is tangent to the line $\varepsilon \cdot v(\alpha_0) = 0$, at $\varepsilon = 0$.

In order to see that $\mathscr{C}$ has the properties stated in the Theorem, assume that $M(r, u)$ is a minimum value of $F(r, u, \cdot)$, i.e., $v''(\alpha_0) \cdot u_0 > 0$. Figure 3.1 shows a situation in which this case occurs. By using the parametrization $u(\theta) = (\cos\theta, \sin\theta)$, $0 \leqslant \theta < 2\pi$, of S^1 and letting θ_0 be such that $u_0 = u(\theta_0)$, it is easy to see that $[\partial M(r, u(\theta))/\partial\theta]_{(r,\theta)=(0,\theta_0)} = -v_1(\alpha_0)\sin\theta_0 + v_2(\alpha_0)\cos\theta_0 < 0$. So that, in a suitable neighborhood of $(r, \theta) = (0, \theta_0)$, $\partial M(r, u(\theta))/\partial\theta < 0$. This means that if $\bar{\varepsilon} \in \mathscr{C}$, with $\bar{r} = |\bar{\varepsilon}|$ sufficiently small, when ε crosses $\mathscr{C}$ over the circle $(\bar{r}\cos\theta, \bar{r}\sin\theta)$, $0 \leqslant \theta < 2\pi$, in the counterclockwise direction (which in this case corresponds to the increasing direction) $M(\bar{r}, u(\theta))$ decreases with θ. Therefore, the number of solutions of (3.2) near $(0, u_0, \alpha_0)$ changes from zero to two. These two solutions correspond to distinct solutions with phases near α_0.

After a finite number of application of the arguments above, it follows from the compactness of the curve $v(\alpha)$, $0 \leqslant \alpha < T$, that a complete description of the diagram of bifurcation can be obtained.

4. PROOF OF THE THEOREM 2 AND COROLLARY

By reason of the nonexistence of α, $0 \leqslant \alpha < T$, for which $v(\alpha)$ and $v'(\alpha)$ are collinear, the curve $v(\alpha)$, $0 \leqslant \alpha < T$, must encircle the origin. Therefore, this hypothesis also imply that, for any $u_0 \in S^1$ there exist precisely two distinct valus α_{01}, α_{02}, $0 \leqslant \alpha_{01}$, $\alpha_{02} < T$, such that $v(\alpha_{01})$, $v(\alpha_{02})$ are both orthogonal to u_0.

The noncollinearity of $v(\alpha_{0j})$ and $v'(\alpha_{0j})$, $j = 1, 2$, allows the use of the implicit function theorem in the same fashion as the first case in the proof of the theorem 1, and conclude that, for each r in some interval $[0, r(u_0)]$ and each u in an open arc around u_0, there exists a unique $\alpha_j^*(r, u)$ near α_{0j}, with α_j^* being a function of class C^1, $\alpha_j^*(0, u_0) = \alpha_{0j}$, $F(r, u, \alpha_j^*(r, u)) = 0$, $j = 1, 2$.

The compactness of S^1 and the uniqueness of the functions α_j^*, $j = 1, 2$, imply after a finite number of applications of the above reasoning that there exists an $\bar{r}_0 > 0$ such that to each pair $(r, u) \in [0, \bar{r}_0) \times S^1$ there corresponds a unique pair $\alpha_j^* = \alpha_j^*(r, u)$, $0 \leqslant \alpha_j^* < T$, $v(\alpha_j^*(0, u)) \cdot u = 0$, $j = 1, 2$, such that $F(r, u, \alpha_1^*(r, u)) = F(r, u, \alpha_2^*(r, u))$.

This means that, given $\varepsilon \neq 0$ in some neighborhood V of $\varepsilon = 0$, there corresponds, via the relation $\varepsilon = ru$, a unique pair of solutions of (1.2), with phases near α_0 and α_1, where α_0, α_1 are given by $v(\alpha_0) \cdot \varepsilon = v(\alpha_1) \cdot \varepsilon = 0$.

Proof of the corollary. By the choice of h_1, its mean value $(1/T) \int_0^T h_1(t)\, dt$ is zero, so that the T-periodic function v_1 is given by

$$v_1(\alpha) = d \int_0^T h_1(t + \alpha)\, r_1(t)\, dt = \left(a_{-1} d \int_0^T e^{-i\omega t} r_1(t)\, dt \right) e^{-i\omega\alpha}$$

$$+ \left(a_1 d \int_0^T e^{i\omega t} r_1(t)\, dt \right) e^{i\omega\alpha} = \cos \omega\alpha.$$

Similarly, one can show that $v_2(\alpha) = \sin \omega\alpha$ and, therefore $v(\alpha)$ is a circle with center $\varepsilon = 0$. The conclusions follow now immediately from the Theorem 2.

5. Proof of the Theorem 3 and Supplementary Remarks

The set $\mathscr{S}$ is contained in the neighborhood $U \subset X$ and both the families, $x(\cdot\,; \beta)$ and $\dot{x}(\cdot\,; \beta)$, $0 < \beta \leqslant 1$, are therefore uniformly bounded. Since $\partial f/\partial x$ is bounded in W, the tubular neighborhood of Γ, it follows that the derivatives of these families are uniformly bounded. The precompactness of $\mathscr{S}$ is now an immediate consequence of the Ascoli–Arzela theorem.

To prove the last part of the theorem it suffices to notice that there exists a continuous function $\alpha(\beta)$ such that the representation (2.2) for $x(\cdot\,; \beta)$ is

$$x(t; \beta) = e(t - \alpha(\beta)) + z(t - \alpha(\beta))$$

$0 \leqslant t < T$, $0 \leqslant \beta < 1$. So that the pair $(\varepsilon(\beta), \alpha(\beta))$ must satisfy the bifurcation equation, i.e.,

$$v(\alpha(\beta)) \cdot \varepsilon(\beta) + S(\varepsilon(\beta), \alpha(\beta)) = 0$$

$0 \leqslant \beta \leqslant 1$. All of the remaining assertions of the theorem follow now from this last equation, the argument is similar to the proof of the Theorem 21 in [8]. This completes, therefore, the proof of Theorem 3.

Supplementary Remarks. Theorem 3 points out a special feature of this problem. Since the bifurcations occur near a family of solutions e_α, $0 \leqslant \alpha < T$, of Eq. (1.1), the solutions do not approach, generally, a particular member of this family, as $\varepsilon \to 0$, but a continuum subfamily. Actually, Theorem 3 gives an indication on how $\varepsilon(\beta)$ should approach zero from S, in order to have the solution $x(\cdot, \beta)$ approach a particular solution of (1.1). It should be along a curve γ: $\varepsilon = \varepsilon(\beta)$ approaching zero in such a way that $\underline{\theta} = \bar{\theta}$ or, equivalently, that either $\varepsilon_1(\beta)/\varepsilon_2(\beta)$ or $\varepsilon_2(\beta)/\varepsilon_1(\beta)$ converge to some limit as $\beta \to 0$, cf. [8, Corollary 1.1] or [2, Chap. 11, Corollary 2.6].

For $|\varepsilon| > 0$ sufficiently small, Eq. (1.2) can have many periodic solutions, with orbit near Γ and the period a rational multiple of T. In fact, there are periodic orbits γ of (1.1) arbitrarily near Γ, with period rationally dependent on T. Since these orbits and f have a common period and the previous results do not depend on T being the minimum period, it follows that everything can be repeated around each γ. Thus, a neighborhood $V(\gamma)$ of $\varepsilon = 0$ can be found in such a way that, for ε in some sectorial subset $S(\gamma)$ of $V(\gamma)$, there are at least two periodic solutions of (1.2). In cases where any finite family of sectors $S(\gamma)$ has a non-empty intersection (when $S(\gamma) = V(\gamma)$, for instance), there exists an arbitrarily large number of periodic (but not T-periodic) solutions of (1.2) under the given hypothesis. However, the existence of an infinity of such solutions can not be ensured, because it depends on uniform estimates on the size of the neighborhoods $V(\gamma)$.

The stability of the solutions of (1.2) in U, can be investigated in the line of results of de Oliveira and Hale [4]. A difficulty in this approach is determined by the fact that both multipliers of the orbit Γ of (1.1) equal one. In [2, Chap. 11, Theorem 2.9], Chow and Hale obtain some results in this direction, dealing with a second-order equation periodically forced and considering subharmonics.

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